PROBLEM SET 8

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Problem 1. *A distribution F is called harmonic if $\sum_1^n \partial_j^2 F = 0$. If F is harmonic and tempered, prove that F is a polynomial.*

Proof. Notice that Fourier transform of a polynomial is a linear combination of Dirac function and its derivatives, it suffices to show $\hat{F} = \sum_{|\alpha|\le N} c_\alpha \partial^\alpha \delta$ for some constants $c_\alpha, |\alpha| \le N$.

Since $\sum \partial_j^2 F = 0$, we have $2\pi i|\xi|^2 \hat{F} = 0$, so either $\hat{F} = 0$ in which case we are done, or $\operatorname{supp} \hat{F} = \{0\}$. Now suppose $\operatorname{supp} \hat{F} = \{0\}$, pick a bump function $\psi \in C_c^\infty$ with $\psi(x) = 1$ for $|x| \le 1/2$ and $\psi(x) = 0$ for $|x| \ge 1$. Then on the one hand by continuity of $\hat{F}$, there exists m, N, C such that

$$|\langle \hat{F}, \phi\rangle| \le C \sum_{|\alpha|\le N} \|\phi\|_{(m,\alpha)}$$

for all $\phi \in C_c^\infty$. On the other hand since $\operatorname{supp} \hat{F} = 0$, we have $\langle \hat{F}, \phi\rangle = \langle \psi \hat{F}, \phi\rangle = \langle F, \psi\phi\rangle$, so (by enlarging constant C if necessary)

$$\begin{aligned}
|\langle \hat{F}, \phi\rangle| &\le C \sum_{|\alpha|\le N} \|\psi\phi\|_{(m,\alpha)} = C \sum_{|\alpha|\le N} \sup_{|x|\le 1} (1+|x|)^m |\partial^\alpha(\psi\phi)(x)| \\
&\le C \sum_{|\alpha|\le N} \sum_{\beta+\gamma=\alpha} \sup_{|x|\le 1} |\partial^\beta \psi(x)||\partial^\gamma \phi(x)| \\
&\le C \sum_{|\alpha|\le N} \sup_{|x|\le 1} |\partial^\alpha \phi(x)|.
\end{aligned}$$

We claim $\langle \hat{F}, \phi\rangle = \langle \hat{F}, T_N\phi\rangle$ for all $\phi \in C_c^\infty$, where $T_N\phi$ is the Taylor series of ϕ at 0 up to degree N. Indeed, let $\eta = \phi - T_N\phi$, then $\partial^\alpha \eta(0) = 0$ for $|\alpha| \le N$. Consider $\eta_k(x) = \eta(x)(1 - \psi(kx))$, then for $|\alpha| \le N$ we have (again C is a positive constant subject to change),

$$\begin{aligned}
\|\partial^\alpha \eta - \partial^\alpha \eta_k\|_u &= \sup_{|x|\le 1/k} |\partial^\alpha(\eta(x)\psi(kx))| \\
&\le C \sum_{\beta+\gamma=\alpha} \sup_{|x|\le 1/k} |(\partial^\beta \eta)(x)| \cdot k^{|\gamma|} \sup_{|x|\le 1/k} |(\partial^\gamma \psi)(kx)| \\
&\le C \sum_{\beta+\gamma=\alpha} \sup_{|x|\le 1/k} |x|^{N+1-|\beta|} \cdot k^{|\gamma|} \|\partial^\gamma \psi\|_u \text{ by Taylor's theorem for } k \text{ big} \\
&\le \frac{C}{k} \sum_{|\gamma|\le N} \|\partial^\gamma \psi\|_u \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

It the follows that $|\langle \hat{F}, \eta - \eta_k\rangle| \leq C \sum\limits_{|\alpha|\leq N} \sup_{|x|\leq 1} |\partial^\alpha \eta - \partial^\alpha \eta_k(x)| \to 0$ as $k \to \infty$, hence $\langle \hat{F}, \eta\rangle = \lim\limits_{k\to\infty} \langle \hat{F}, \eta_k\rangle = 0$ since η_k is supported away from 0. Therefore

$$\langle \hat{F}, \phi\rangle = \langle \hat{F}, T_N\phi + \eta\rangle = \langle \hat{F}, T_N\phi\rangle.$$

This implies, for all $\phi \in C_c^\infty$, we have

$$\begin{aligned}\langle \hat{F}, \phi\rangle &= \langle \hat{F}, \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\partial^\alpha\phi(0)x^\alpha\rangle = \sum_{|\alpha|\leq N} \frac{\langle \hat{F}, x^\alpha\rangle}{\alpha!}\partial^\alpha\phi(0) \\ &= \langle \sum_{|\alpha|\leq N} (-1)^{|\alpha|} \frac{\langle \hat{F}, x^\alpha\rangle}{\alpha!}\partial^\alpha\delta, \phi\rangle.\end{aligned}$$

Hence $\hat{F} = \sum\limits_{|\alpha|\leq N} (-1)^{|\alpha|} \frac{\langle \hat{F}, x^\alpha\rangle}{\alpha!}\partial^\alpha\delta$ and F is a polynomial. □

Problem 2. *Let f be a continuous function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree $-n$, $f(rx) = r^{-n}f(x)$ for $r > 0$, and has mean 0 on the unit sphere. Show that f is not locally integrable near 0 unless $f = 0$, but f defines a tempered distribution $PV(f)$ by*

$$\langle PV(f), \phi\rangle = \lim_{\epsilon\to 0} \int_{|x|>\epsilon} f(x)\phi(x)dx.$$

The limit equals

$$\int_{|x|\leq 1} f(x)[\phi(x) - \phi(0)]dx + \int_{|x|>1} f(x)\phi(x)dx,$$

and these integrals converge absolutely.

Proof. We show f is locally integrable near 0 unless $f = 0$. Indeed for $\epsilon > 0$ we have

$$\int_{|x|<\epsilon} |f(x)|dx = \int_0^\epsilon \int_{S^{n-1}} |f(ry)|r^{n-1}d\sigma(y)dr = \int_0^\epsilon \frac{dr}{r} \int_{S^{n-1}} |f(y)|d\sigma(y)$$

is finite unless $f = 0$. Now fix $\epsilon < 1$, we have

$$\begin{aligned}|\int_{\epsilon<|x|\leq 1} f(x)\phi(x)dx| &= |\int_{\epsilon<|x|\leq 1} f(x)(\phi(x) - \phi(0))dx| \\ &\leq \int_{S^{n-1}} |f(y)| \int_\epsilon^1 |\frac{\phi(ry) - \phi(0)}{r}|drd\sigma(y) \\ &\leq (1-\epsilon)\|f\|_{L^1(S^{n-1})}\|\phi\|_{H_1(\mathbb{R}^n)} < \infty\end{aligned}$$

where the first equality follows easily from f is homogeneous with mean 0 on unit sphere. This proves $\int\limits_{\epsilon<|x|\leq 1} f(x)\phi(x)dx = \int\limits_{\epsilon<|x|\leq 1} f(x)[]\phi(x) - \phi(0)]dx$ absolutely converges as $\epsilon \to 0$. On the other hand, we have

$$\begin{aligned}|\int_{|x|>1} f(x)\phi(x)dx| &= \int_1^\infty \int_{S^{n-1}} |f(y)|\frac{|\phi(ry)|}{r}drd\sigma(y) \\ &\leq \|f\|_{L^1(S^{n-1})}\|\phi\|_{N,0} \int_1^\infty \frac{dr}{r(1+r)^N} < \infty \quad \text{for some big } N.\end{aligned}$$

So $\lim_{\epsilon\to 0}\int_{|x|>\epsilon} f(x)\phi(x)dx$ exists as $\int\limits_{|x|\le 1} f(x)[\phi(x)-\phi(0)]dx + \int\limits_{|x|>1} f(x)\phi(x)dx$. □

Problem 3. *On $\mathbb{R}$, let $F = PV((\pi x)^{-1})$. Check that*

1. *$\hat{F}(\xi) = -i\,\mathrm{sgn}\,\xi$.*
2. *The map $\phi \to F * \phi$, initially defined on Schwartz functions, extends to a unitary operator on L^2.*

This is called the Hilbert transform.

Proof. (1) We calculate by definition,

$$\begin{aligned}\hat{F}(\xi) &= \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{1}{\pi x}e^{-2\pi i x\xi}dx \\ &= -\frac{i\,\mathrm{sgn}(\xi)}{\pi}\lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\sin(2\pi x|\xi|)}{x}dx \quad \text{since } \cos(2\pi x\xi)/x \text{ is odd and sin is odd} \\ &= -\frac{i\,\mathrm{sgn}(\xi)}{\pi}\lim_{\epsilon\to 0}\int_{|t|\ge 2\pi|\xi|\epsilon}\frac{\sin t}{t}dt \quad (t = 2\pi|\xi|x) \\ &= -i\,\mathrm{sgn}(\xi).\end{aligned}$$

(2) Notice $|\hat{F}| = 1$, so for $\phi, \psi \in \mathcal{S}$ we have

$$\langle F*\phi, F*\psi\rangle_2 = \langle \hat{F}\hat{\phi}, \hat{F}\hat{\psi}\rangle_2 = \langle |\hat{F}|^2\hat{\phi}, \hat{\psi}\rangle_2 = \langle\hat{\phi}, \hat{\psi}\rangle_2 = \langle\phi,\psi\rangle_2.$$

So $\phi \to F*\phi$ is a unitary operator on $\mathcal{S}$ under L^2 inner product, and naturally extends to a unitary operator on L^2 since $\mathcal{S}$ is dense in L^2 under L^2-norm.

□

Problem 4. *Give $C^\infty(\mathbb{T}^n)$ the Fréchet space topology defined by the seminorms $\|\phi\|_{(\alpha)} = \|\partial^\alpha\phi\|_\infty$. The space $\mathcal{D}'(\mathbb{T}^n)$ of distributions on $\mathbb{T}^n$ is the space of continuous linear functionals on $C^\infty(\mathbb{T}^n)$, with the weak * topology. Prove the following.*

1. *Distributions on $\mathbb{T}^n$ can be translated, differentiated, and multiplied by C^∞ functions, just as on $\mathbb{R}^n$.*
2. *If $F \in \mathcal{D}'(\mathbb{T}^n)$, its Fourier transform is the function $\hat{F}$ on $\mathbb{Z}^n$ defined by $\hat{F}(\kappa) = \langle F, E_\kappa\rangle$ where $E_\kappa(x) = e^{-2\pi i\kappa\cdot x}$. Prove that a function g on $\mathbb{Z}^n$ is the Fourier transform of a distribution on $\mathbb{T}^n$ iff $|g(\kappa)| \le C(1+|\kappa|)^N$ for some $C, N > 0$.*
3. *If $F \in \mathcal{D}'(\mathbb{T}^n)$ and $\phi \in C^\infty(\mathbb{T}^n)$, prove $\langle F, \overline{\phi}\rangle = \sum_\kappa \hat{F}(\kappa)\overline{\hat{\phi}(\kappa)}$.*

Proof. (1) We may treat functions on $\mathbb{T}^n$ as n-periodic functions on $\mathbb{R}^n$ and translation, differentiation, multiplication by smooth functions are defined on functions, hence one can dualize those operation to distributions.

(2) The only if part easily follows from the continuity of $F \in \mathcal{D}'(\mathbb{T}^n)$. To elaborate, we may find C, N so that $|\langle F, \phi\rangle| \le C\sum_{|\alpha|\le N}\|\phi\|_{(\alpha)}$, in particular

$$|\hat{F}(\kappa)| \le C\sum_{|\alpha|\le N}\sup_{x\in\mathbb{T}^n}|\partial^\alpha E_k(x)| \le C'(1+|\kappa|)^N.$$

Now suppose g is a function on $\mathbb{Z}^n$ satisfying $|g(\kappa)| \le C(1+|\kappa|)^N$ for some $C, N > 0$. Consider $S_m g(x) = \sum_{|\kappa|\le m} g(\kappa)e^{2\pi i\kappa\cdot x}$. It is clear that $S_m g \in C^\infty(\mathbb{T}^n)$, and $\lim\limits_{m\to\infty} S_m g$ is the Fourier inversion of g if the limit exists, thus it remains to show $S_m g$ converges to a distribution as $m \to \infty$. For any $\phi \in C^\infty(\mathbb{T}^n)$, we have

$$\int_{\mathbb{T}^n} S_m g(x)\phi(x)dx = \sum_{|\kappa|\le m} g(\kappa)\int_{\mathbb{T}^n}\phi(x)e^{2\pi i\kappa\cdot x}dx = \sum_{|\kappa|\le m} g(\kappa)\hat{\phi}(-\kappa).$$

Since $|g(\kappa)| \le C(1+|\kappa|)^N$ whilst ϕ is smooth so that $|\hat{\phi}(-\kappa)|$ decays rapidly than any polynomial, we can find constant C' so that $|g(\kappa)\hat{\phi}(-\kappa)| \le C'(1+|\kappa|^{-n-1})$ for all $\kappa \in \mathbb{T}^n$. This proves $\sum_{|\kappa|\le m} g(\kappa)\hat{\phi}(-\kappa)$ is absolutely convergent as $m \to \infty$, thus $\lim\limits_{m\to\infty} S_m g$ exists.

(3) Replacing ϕ by $\bar{\phi}$ and g by $\hat{F}$, the identity follows from almost exactly the same calculation in (2).

□

Problem 5. *If $k \in \mathbb{N}$, H_k is the space of L^2 functions whose L^2 derivatives up to order k exist. Show that these strong derivatives coincide with the distribution derivatives.*

Proof. Without loss of generality, we prove the case where $k = 1, n = 1$. Let $f \in H_1$. For any $\phi \in C_c^\infty$ and $y > 0$, then by an easy change of variable we have

$$\langle \frac{f(x+y)-f(x)}{y}, \phi\rangle_{L^2_x} = \int \frac{f(x+y)-f(x)}{y}\phi(x)dx = \int f(x)\frac{\phi(x-y)-\phi(x)}{y}dx$$
$$= \langle f, \frac{\phi(x-y)-\phi(x)}{y}\rangle_{L^2_x}.$$

Now since $f \in H^1$, $\phi \in C_c^\infty$, $\frac{f(x+y)-f(x)}{y}$, resp. $\frac{\phi(x-y)-\phi(x)}{y}$, converges to $f'(x)$ resp. $-\phi'(x)$ in L^2 as $y \to 0$. So taking limit in the above identity, we have $\langle f', \phi\rangle = -\langle f, \phi'\rangle$. This proves the strong derivative of f is its distributional derivative. □